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Simulation of Dimensionally Reduced SYM–Chern–Simons Theory

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Abstract

A supersymmetric formulation of a three-dimensional SYM–Chern–Simons theory using light-cone quantization is presented, and the supercharges are calculated in light-cone gauge. The theory is dimensionally reduced by requiring all fields to be independent of the transverse dimension. The result is a non-trivial two-dimensional supersymmetric theory with an adjoint scalar and an adjoint fermion. We perform a numerical simulation of this SYM–Chern–Simons theory in 1+1 dimensions using SDLCQ (Supersymmetric Discrete Light-Cone Quantization). We find that the character of the bound states of this theory is very different from previously considered two-dimensional supersymmetric gauge theories. The low-energy bound states of this theory are very "QCD-like." The wave functions of some of the low mass states have a striking valence structure. We present the valence and sea parton structure functions of these states. In addition, we identify BPS-like states which are almost independent of the coupling. Their masses are proportional to their parton number in the large-coupling limit.

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I. INTRODUCTION

Chern–Simons (CS) theories are certainly some of the most interesting field theories in 2+1 dimensions. Among the interesting phenomena one sees in these theories are: the quantum hall effect, Landau levels, non-trivial topological structures, vortices, and anyons. For a review of these phenomena see [1]. Some of these (2+1)-dimensional phenomena have been observed experimentally in condensed matter systems. CS theories are also often described as topological theories [2] since for most gauge groups the CS coupling must obey a quantization condition for the theory to remain gauge invariant. Within this rich literature on CS theories there is also considerable work on SYM–CS theories. These theories have their own remarkable properties. It has been shown that there is a finite anomaly that shifts the CS coupling [3], and it has been conjectured by Witten [4] that this theory spontaneously breaks supersymmetry for some values of the CS coupling. There are other related reasons for interest in CS theories. Witten [5] has conjectured that string field theory is essentially a non-commutative CS theory. Recently this led to a conjecture by Susskind [6] that relates string theory to the fractional quantum Hall effect.

Since these theories are interesting from so many different points of view, it is certainly useful to numerically simulate them. The method we will use is SDLCQ (Supersymmetric Discrete Light-Cone Quantization). This is a numerical method that can be used to solve any theory with enough supersymmetry to be finite. The central point of this method is that using DLCQ we can construct a finite dimensional representation of the superalgebra [7]. From this representation of the superalgebra, we construct a finite-dimensional Hamiltonian which we diagonalize numerically. We repeat the process for larger and larger representations and extrapolate the solution to the continuum. We have already solved standard (2+1)-dimensional supersymmetric Yang–Mills (SYM) theories by this method [8,9], so it is clear that SYM-CS theories are within our reach.

In this paper we will start with SYM–CS theory in 2+1 dimensions and dimensionally reduce it to two dimensions by requiring all of the fields to be independent of the transverse coordinate. This is an interesting (1+1)-dimensional supersymmetric theory in its own right. It is a good starting point for our numerical simulations because we can solve this problem using two completely independent codes, one a MATHEMATICA code and the other a C++ code. When we move on to 2+1 dimensions in future work, the (1+1)-dimensional results will be related to the (2+1)-dimensional results in a nontrivial way. We found this to be the case in our previous work on (2+1)-dimensional SYM theories, and it will most probably be the case here as well. This will provide us an important check on our code for solving the (2+1)-dimensional problem. It is important to develop this chain of numerical checks, since there are no analytical or other numerical solutions to check our results against.

Many of the most interesting aspects of CS theory will be lost by this reduction, most notably the quantization of the CS coupling. However, one particularly interesting property that will be preserved is the fact that the CS term simulates a mass for the theory. It is well known that supersymmetric abelian CS theory is simply the theory of a free massive fermion and a free massive boson. In the non-abelian theory additional interactions are introduced, but we expect to also see this mass effect. This is particularly interesting here because dimensionally reduced $\mathcal{N}=1$ SYM is a very stringy theory. The low-mass states are dominated by Fock states with many constituents, and as the size of the superalgebraic representation is increased, states with lower masses and more constituents appear [8–20]. The connection between string theory and supersymmetric gauge theory leads one to expect this type of behavior; however, these gauge theories are not very QCD-like. Ultimately one might like to make a connection with the low-mass spectrum observed in nature.

We expect to find states in SYM–CS theory that are much more QCD-like, because of the effective mass for the constituents introduced by the CS term. We will see that some of the low-mass states will be dominated by valence-like Fock states that have only a few constituents. As we go to larger and larger representation of the superalgebra, we will resolve these states better and better, and any new states that appear will be heavier. We will also find highly mixed states without a valence structure. Finally, we find that at strong coupling the low-energy spectrum is dominated by states that are a reflection of the BPS states of the SYM theory in 1+1 dimensions [20] and are therefore independent of the coupling g.

¿From the wave functions of the bound states we will be able to find the structure functions of both the valence and sea partons. Some of the highly mixed states will have a double-humped structure function. Some of these structure functions are similar to those conjectured in various phenomenological calculations; they are the result of the solution of a nontrivial gauge theory.

In Sec. II we give a general discussion of supersymmetric light-cone-quantized SYM–CS theory in light-cone gauge. We then give the dimensionally reduced discrete formulation of the supercharges and discuss the other symmetries of the theory. In Sec. III we discuss our numerical method and some of the new wrinkles that appear in CS theory. We also present and discuss in this section the spectrum of bound states as well as a variety of properties of these states including their structure functions. We will see that it is a very QCD-like theory as opposed to pure SYM theory, which is very stringy. Finally, in Sec. IV, we summarize the results and discuss the prospects and challenges for calculating in the full (2+1)-dimensional theory.

II. SUPERSYMMETRIC CHERN–SIMONS THEORY

We will consider $\mathcal{N} = 1$ supersymmetric CS theory in 2+1 dimensions as the starting point of our discussion. While we will reduce this theory to a (1+1)-dimensional theory for numerical simulation here, we eventually plan to simulate the full theory, and it is therefore useful to present a detailed light-cone formulation in light-cone gauge. The Lagrangian of this theory is

$$\mathcal{L} = \mathrm{Tr}(-\frac{1}{4}\mathcal{L}_{\mathrm{YM}} + i\mathcal{L}_{\mathrm{F}} + \frac{\kappa}{2}\mathcal{L}_{\mathrm{CS}}), \tag{2.1}$$

where κ is the CS coupling and

$$\mathcal{L}_{\mathrm{YM}} = F_{\mu\nu}F^{\mu\nu}\,, \tag{2.2}$$

$$\mathcal{L}_{\mathrm{F}} = \bar{\Psi}\gamma_\mu D^\mu\Psi\,, \tag{2.3}$$

$$\mathcal{L}_{\mathrm{CS}} = \epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}gA_\mu A_\nu A_\lambda\right) + 2\bar{\Psi}\Psi\,. \tag{2.4}$$

The two components of the spinor $\Psi = 2^{-1/4}\binom{\psi}{\chi}$ are in the adjoint representation of $U(N_c)$ or $SU(N_c)$. We will work in the large-N_c limit. The field strength and the covariant derivative are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\,, \qquad D_\mu = \partial_\mu + ig[A_\mu, \quad]\,. \tag{2.5}$$

The supersymmetric variations of the fields are

$$\delta A_\mu = i\bar{\epsilon}\gamma_\mu \Psi\,, \tag{2.6}$$

$$\delta\Psi = \frac{1}{4} i\epsilon^{\mu\nu\lambda}\gamma_\lambda F_{\mu\nu} = \frac{1}{4}\Gamma^{\mu\nu}\epsilon F_{\mu\nu}\,, \tag{2.7}$$

where[1]

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_3, \quad \Gamma^{\mu\nu} \equiv \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = i\epsilon^{\mu\nu\lambda}\gamma_\lambda\,. \tag{2.8}$$

This leads to the supercurrent $Q^{(\mu)}$ in the usual manner via

$$\delta\mathcal{L} = \bar{\epsilon}\partial_\mu Q^{(\mu)}. \tag{2.9}$$

Light-cone coordinates in 2+1 dimensions are $(x^+, x^-, x^\perp)$ where $x^+ = x_-$ is the light-cone time and $x^\perp = -x_\perp$. The totally anti-symmetric tensor is defined by $\epsilon^{+-2} = -1$. The variations of the three parts of the Lagrangian in Eq. (2.1) determine the ('chiral') components $Q^\pm$ of the supercharge via Eq. (2.9) to be

$$\int d^2x Q^{(+)} = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix} = \frac{i}{2}\int d^2x\, \Gamma^{\alpha\beta}\gamma^+\Psi F_{\alpha\beta}\,. \tag{2.10}$$

Explicitly they are

$$\begin{aligned} Q^- &= -i2^{3/4}\int d^2x\,\psi\left(\partial^+A^- - \partial^-A^+ + ig[A^+, A^-]\right)\,, \\ Q^+ &= -i2^{5/4}\int d^2x\,\psi\left(\partial^+A^2 - \partial^2A^+ + ig[A^+, A^2]\right)\,. \end{aligned} \tag{2.11}$$

One can convince oneself by calculating the energy-momentum tensor $T^{\mu\nu}$ that the supercharge fulfills the supersymmetry algebra

$$\{Q^\pm, Q^\pm\} = 2\sqrt{2}P^\pm\,, \qquad \{Q^+, Q^-\} = -4P^\perp\,. \tag{2.12}$$

In order to express the supercharge in terms of the physical degrees of freedom, we have to use equations of motion, some of which are constraint equations. The equations of motion for the gauge fields are

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[1] This choice of representation for the Dirac matrices is different from that of Ref. [1] by the interchange of γ^1 and γ^2 but is more natural for light-cone quantization. In this representation the spinor term of the CS Lagrangian enters Eq. (2.4) with a plus sign.

$$D_\nu F^{\nu\alpha} = -J^\alpha\,, \tag{2.13}$$

where

$$J^\alpha = \frac{\kappa}{2}\epsilon^{\alpha\nu\lambda}F_{\nu\lambda} + 2g\bar{\Psi}\gamma^\beta\Psi\,. \tag{2.14}$$

For $\alpha = +$ this is a constraint for A^-,

$$D_- A^- = -(D_2-\kappa)A^2 - \frac{1}{D_-}(D_2-\kappa)\partial_2 A^+ + 2g\frac{1}{D_-}\bar{\Psi}\gamma^+\Psi\,. \tag{2.15}$$

In light-cone gauge, $A^+=0$, this reduces to

$$D_- A^- = \frac{1}{D_-}[(\kappa - D_2)D_- A^2 + 2g\bar{\Psi}\gamma^+\Psi]\,. \tag{2.16}$$

The equation of motion for the fermion is

$$\gamma^\mu D_\mu \Psi = -i\kappa\Psi\,. \tag{2.17}$$

Expressing everything in terms of ψ and χ leads to the equations of motion

$$\sqrt{2}D_+\psi = (D_2+\kappa)\chi\,, \tag{2.18}$$
$$\sqrt{2}D_-\chi = (D_2-\kappa)\psi\,, \tag{2.19}$$

the second of which is a constraint equation. The constraint equations are used to eliminate χ and A^-.

We now reduce the theory dimensionally to two dimensions by setting $\phi = A_2$ and $\partial_2 \to 0$ for all fields. This yields, from Eq. (2.11) and the constraints,

$$Q^- = 2^{3/4} g \int dx^- \left(i[\phi, \partial_-\phi] + 2\psi\psi - \frac{\kappa}{g}\partial_-\phi \right) \frac{1}{\partial_-}\psi\,. \tag{2.20}$$

The mode expansions in two dimensions are

$$\phi_{ij}(0,x^-) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{dk^+}{\sqrt{2k^+}}\left[a_{ij}(k^+)e^{-\mathrm{i}k^+x^-} + a^\dagger_{ji}(k^+)e^{\mathrm{i}k^+x^-}\right]\,,$$
$$\psi_{ij}(0,x^-) = \frac{1}{2\sqrt{\pi}}\int_0^\infty dk^+\left[b_{ij}(k^+)e^{-\mathrm{i}k^+x^-} + b^\dagger_{ji}(k^+)e^{\mathrm{i}k^+x^-}\right]\,. \tag{2.21}$$

To discretize the theory we impose periodic boundary conditions on the boson and fermion fields alike, and obtain an expansion of the fields with discrete momentum modes. Thus the discrete version of the CS part of the supercharge is

$$Q^-_{CS} = \left(\frac{ig2^{-1/4}\sqrt{L}}{\pi}\right)(-ih)\sum_n \frac{1}{\sqrt{n}}\left(A^\dagger(n)B(n) + B^\dagger(n)A(n)\right)\,, \tag{2.22}$$

where $h \equiv \sqrt{\pi}\kappa/g$ is a rescaled CS coupling and A and B are rescaled discrete field operators

$$A(n) \equiv \sqrt{\frac{\pi}{L}} a_{ij}(n\pi/L)\,, \qquad B(n) \equiv \sqrt{\frac{\pi}{L}} b_{ij}(n\pi/L)\,. \tag{2.23}$$

The ordinary supersymmetric part of the supercharge is listed elsewhere [8]. It is important to note that the supercharge for $\mathcal{N}=1$ SYM in 2+1 dimensions has a contribution of the form

$$Q_\perp^- = \left(\frac{ig2^{-1/4}\sqrt{L}}{\pi}\right) \sum_{n,n_\perp} \frac{n_\perp}{\sqrt{n}} \left(A^\dagger(n,n_\perp)B(n,n_\perp) - B^\dagger(n,n_\perp)A(n,n_\perp)\right)\,. \tag{2.24}$$

Of course, the light-cone energy is $(k_\perp^2 + m^2)/k^+$, so $k_\perp$ behaves like a mass, and here we see that h appears in a very similar way to $k_\perp$ and therefore behaves in many ways like a mass.

When comparing the two contributions to the supercharge, we see that we have a relative i between them. Thus the usual eigenvalue problem

$$2P^+P^-|\varphi\rangle = \sqrt{2}P^+(Q^-)^2|\varphi\rangle = \sqrt{2}P^+(Q^-_{\rm SYM} + Q^-_{\rm CS})^2|\varphi\rangle = M_n^2|\varphi\rangle \tag{2.25}$$

has to be solved by using fully complex methods.

We retain[2] the S-symmetry, which is associated with the orientation of the large-N_c string of partons in a state [21]. In a (1+1)-dimensional model this orientation parity is usually referred as a Z_2 symmetry, and we will follow that here. It gives a sign when the color indices are permuted

$$Z_2 : a_{ij}(k) \to -a_{ji}(k)\,, \qquad b_{ij}(k) \to -b_{ji}(k)\,. \tag{2.26}$$

We will use this symmetry to reduce the Hamiltonian matrix size and hence the numerical effort. All of our states will be labeled by the Z_2 sector in which they appear. We will not attempt to label the states by their normal parity; in the light-cone this is only an approximate symmetry. Such a labeling could be done in an approximate way, as was shown by Hornbostel [22], and might be useful for comparison purposes if at some point there are results from lattice simulations of the present theory.

III. NUMERICAL RESULTS

We convert the mass eigenvalue problem $2P^+P^-|M\rangle = M^2|M\rangle$ to a matrix eigenvalue problem by introducing a basis where P^+ is diagonal. In SDLCQ this is done by first discretizing the supercharge Q^- and then constructing P^- from the square of the supercharge: $P^- = (Q^-)^2/\sqrt{2}$. To discretize the supercharge, we introduce discrete longitudinal momenta k^+ as fractions nP^+/K of the total longitudinal momentum P^+, where K is an integer that determines the resolution of the discretization and is known in DLCQ as the harmonic resolution [23]. Because light-cone longitudinal momenta are always positive, K and each n are positive integers; the number of constituents is then bounded by K. The integrals in Q^-

[2] We note that in three dimensions the CS term breaks transverse parity.

are approximated by a trapezoidal form. The continuum limit is then recovered by taking the limit $K \to \infty$.

In constructing the discrete approximation we drop the longitudinal zero-momentum mode. For some discussion of dynamical and constrained zero modes, see the review [24] and previous work [10]. Inclusion of these modes would be ideal, but the techniques required to include them in a numerical calculation have proved to be difficult to develop, particularly because of nonlinearities. For DLCQ calculations that can be compared with exact solutions, the exclusion of zero modes does not affect the massive spectrum [24]. In scalar theories it has been known for some time that constrained zero modes can give rise to dynamical symmetry breaking [24], and work continues on the role of zero modes and near zero modes in these theories [25].

To obtain the spectrum of the CS theory we solve the complex eigenvalue problem, Eq. (2.25). For the numerical evaluation we can exploit the structure of the supercharge

$$Q^- = \begin{pmatrix} 0 & A+iB \\ A^T - iB^T & 0 \end{pmatrix}, \tag{3.1}$$

where A and B are real matrices. The Hamiltonian has thus an easy decomposition into a real and imaginary part in the bosonic sector

$$P^-_{\rm boson} = AA^T + BB^T + i\left(BA^T - AB^T\right) . \tag{3.2}$$

and the fermionic sector

$$P^-_{\rm fermion} = A^TA + B^TB + i\left(A^TB - B^TA\right) , \tag{3.3}$$

Our earliest SDLCQ calculations were done using a code written in Mathematica and performed on a PC. This code was then rewritten in C++ and substantially revised. It runs on a Linux workstation with 2 GB of RAM and can handle as many as 2,000,000 Fock states. The present calculation was done in both codes as a check. We limit the calculation to resolution $K = 9$ because it seems sufficient here and because the present C++ code, which was primarily written for (2+1)-dimensional models, would require some non-trivial modifications to run at higher longitudinal resolutions.

The low-energy spectrum, with $h = 1.0$, is fit to $M^2 = M^2_\infty + b(1/K)$. These fits are shown in Fig. 1 for some of the low-energy states. In principle, higher energy states can also be found, but at these couplings the states just above these are difficult to disentangle because of level crossings. In the Z_2 even sector we find the continuum masses to be (in units of g^2N_c/π) $M^2_\infty = 4.30$, 18.33, 27.46, and 43.20, whereas in the Z_2 odd sector we have $M^2_\infty = 10.06$, 29.13, 32.83, 39.52, and 47.40.

The CS term in this theory effectively generates a mass proportional to the CS coupling. Therefore, we expect the low-mass states will only have a few partons. This property will be even more apparent as we increase the CS coupling. This is interesting and important for two reasons. First, it stands in stark contrast to $\mathcal{N} = 1$ SYM theory, which is very stringy and has a large number of low-mass states with a large number of partons. Second, this behavior of the CS theory is reminiscent of QCD. In Fig. 2 we plot the spectrum of the theory as a function of the scaled CS coupling h. We see that the masses of the bound states

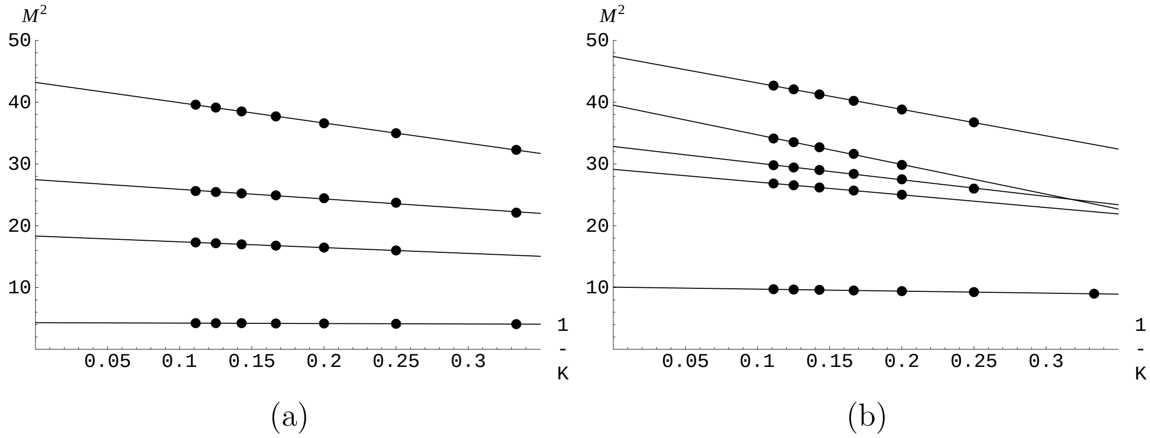


FIG. 1. Low-lying spectrum of the two-dimensional theory in units of g^2N_c/π at Chern–Simons coupling $h = 1.0$ for the (a) Z_2 even sector and (b) Z_2 odd sector.

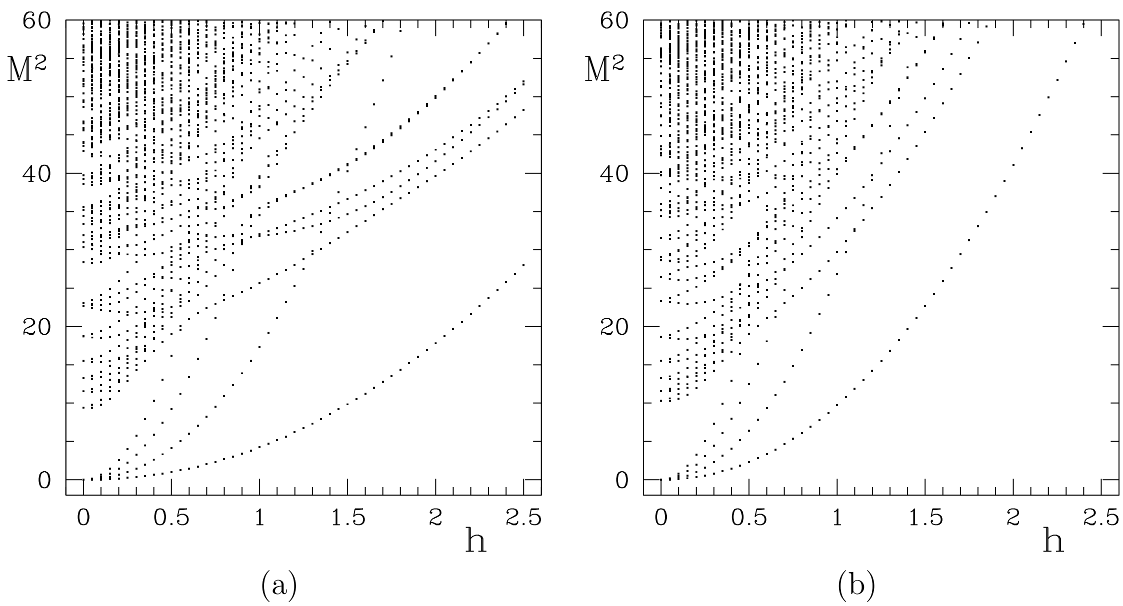
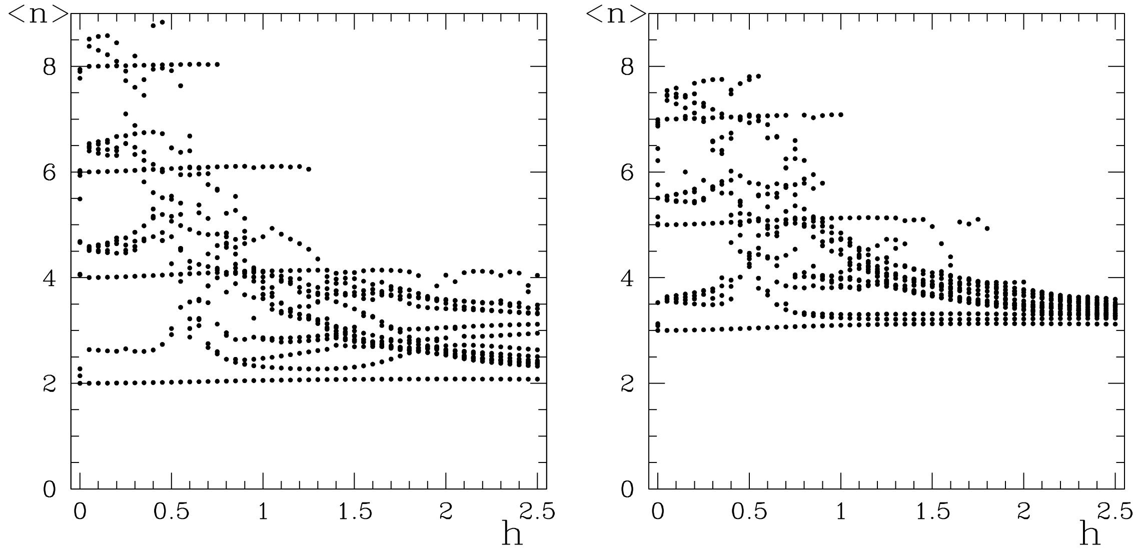


FIG. 2. Bosonic spectrum of the two-dimensional theory in units of g^2N_c/π at $K = 9$ as a function of the Chern–Simons coupling h for the (a) Z_2 even sector and (b) Z_2 odd sector.



(a) (b)

FIG. 3. Average parton number of the 15 lowest bosonic states at $K = 9$ as a function of h for the (a) Z_2 even sector and (b) Z_2 odd sector.

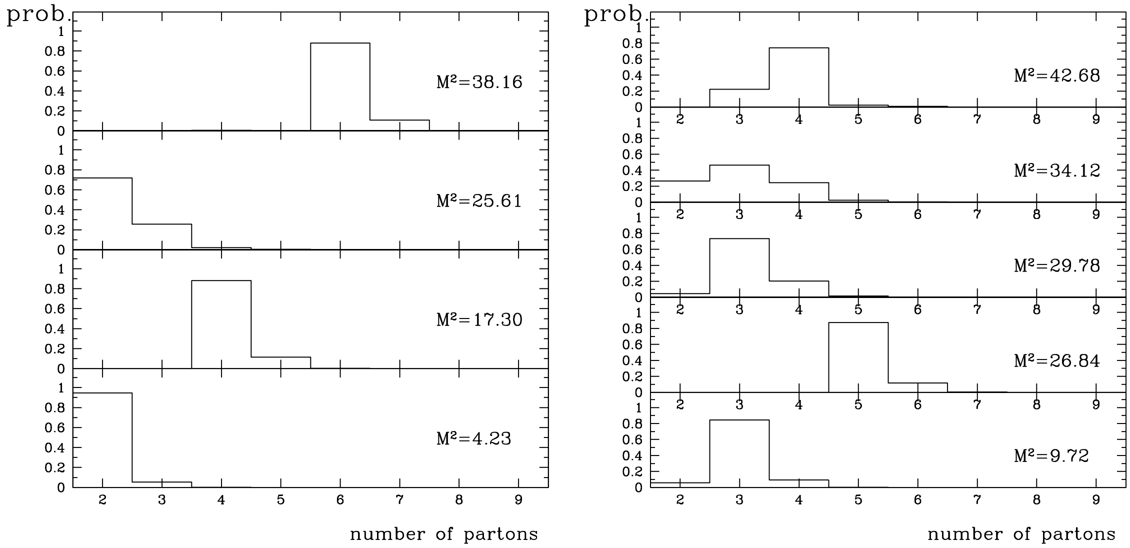
grow with the CS coupling, i.e. the effective mass of the constituents. We also see that there are a lot of level crossings which make it hard to follow the trajectories of individual states.

In Fig. 3 we plot the average parton number in the fifteen lowest states in each sector as a function of the CS coupling for resolution $K = 9$. The average parton numbers range between 2 and the maximally allowed $\langle n \rangle = 9$ at $h = 0$ and decrease to below 4 at $h = 2.5$. The disrupted trajectories $\langle n \rangle(h)$ are, of course, due to the level crossings. Most prominently we have crossings at $h \approx 0.6$ and $h \approx 1.3$, cf. Fig. 2, which are reflected in the discontinuities of the $\langle n \rangle(h)$ trajectories at these points. The apparent lack of states with $\langle n \rangle \approx 2$ for $0 < h < 0.5$ can be explained by the mixing of very light states of very different parton content. At $h = 0$ we have $2(K-1)$ massless states which have parton numbers all the way up to 9. These states mix to give the very light states at $h > 0$, which eventually are distinct enough to form independent $\langle n \rangle(h)$ trajectories.

In Fig. 4 we plot the probability of the nine lowest-energy bound states to have a specific number of partons. In the Z_2 even sector, the two lowest states are nearly pure two-particle and four-particle bound states, respectively, while the higher states shown have mixed content. We have looked at many of the higher mass states, and we find other nearly pure states, but almost always with an even number of partons. Similarly in the Z_2 odd sector we find that the two lowest states are nearly pure but now have three and five partons. Again the high states shown are of mixed parton number, but there are other higher mass states not shown that are nearly pure but almost always have an odd number of partons. The probabilities for the degenerate fermionic bound states are not shown, but they are virtually identical to these in the respective sectors. The fermionic states for the most part just have one of the bosons replaced by a fermion.

The general structure of the supercharge for this CS theory is

$$Q^- = gQ^-_{\rm SYM} + \kappa Q^-_{\rm CS}\,, \tag{3.4}$$

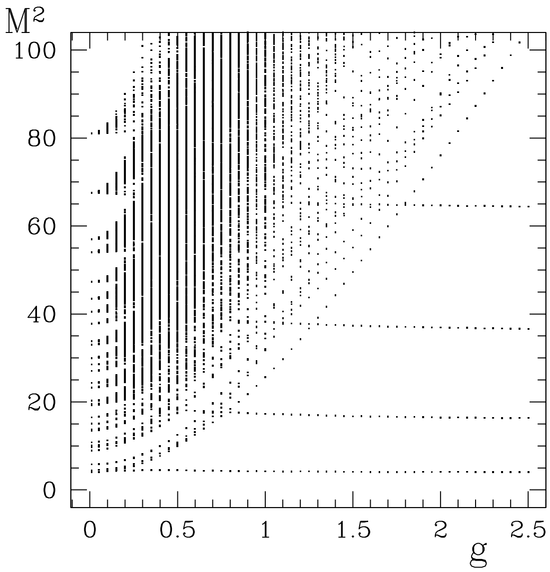


(a) (b)

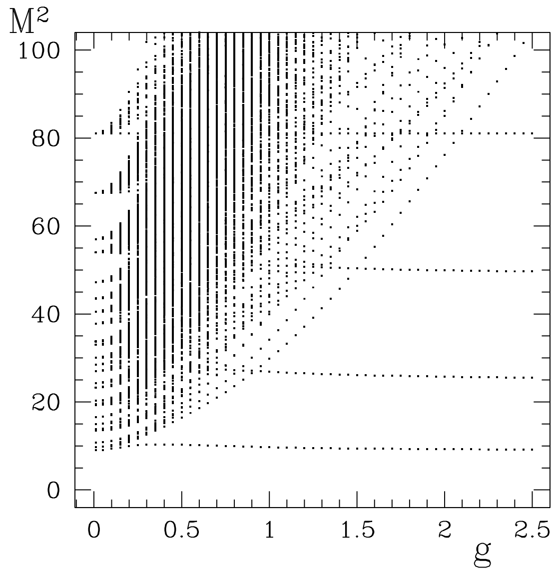
FIG. 4. Parton probability distributions for the 9 states in Fig. 1 for the (a) Z_2 even sector and (b) Z_2 odd sector.

where $Q^-_{\rm SYM}$ is the supercharge of the $\mathcal{N}=1$ SYM theory of adjoint fermions and adjoint bosons, which was studied extensively in [20], and $Q^-_{\rm CS}$ is the contribution of the CS interaction to the supercharge, given in (2.22). The Hamiltonian is the square of the supercharge, and we therefore expect that as a function of g and κ the spectrum of this theory will grow quadratically in both variables. In Fig. 2 we see this behavior as a function of $h=\sqrt{\pi}\kappa/g$ at fixed g. In Fig. 5 we see this general behavior at fixed κ as a function of g as well. There are, however, a number of very special states visible in this figure. These states are reflections of the BPS states that we found in the pure SYM theory [20]. Since the central charge is zero in that theory, the BPS states are exactly massless and are annihilated by the supercharge $Q^-_{\rm SYM}$. The special states that we see in Fig. 5 are essentially these BPS states arranged with a fixed number of particles. Thus the masses of these states are approximately independent of g and proportional to the number of partons squared. In this theory at strong coupling, clearly the low-mass states are these BPS related states. We have already found these BPS states in the (2+1)-dimensional SYM theory [14,16], and, therefore, we expect that these states will dominate the low-energy spectrum of the (2+1)-dimensional CS theory at strong coupling.

We would expect that at $g=0$ all states have masses $M^2=n^2$ in units of $\kappa^2 N_c$, where n is an even(odd) integer in the Z_2 even(odd) sector of the theory. At $g=0$ only the CS part of the supercharge is present, and it is basically a parton number operator. What we find, however, are states with well-defined masses, not only at $M^2=n^2$, but also at intermediate values, which are in general highly degenerate. The degeneracy is lifted as the coupling g is turned on, and in the strong coupling limit all but the BPS states decouple, as can be seen in Fig. 5. A closer inspection of the convergence of the eigenvalues immediately reveals that the states at $g=0$ are actually multi-particle states built out of constituents with "mass" proportional to κ. This is very much like what happens in the large N_f limit of



(a)



(b)

FIG. 5. Bosonic spectrum of the two-dimensional theory in units of $\kappa^2 N_c$ as a function of the gauge coupling g at fixed Chern–Simons coupling κ for the (a) Z_2 even sector and (b) Z_2 odd sector. The values of the gauge coupling are defined in units of $\kappa/\sqrt{\pi}$. The resolution is $K = 9$.

adjoint QCD$_2$ [26]. Actually, the DLCQ spectra $M_i^2(K)$, as a function of resolution K, look almost exactly like the analogous plots in Ref. [26], and we suppress them here. They are consistent with the DLCQ expression for the kinetic energy of free particles. For example, the two-particle formula is

$$\frac{M^2(K)}{K} = \frac{M^2(n)}{n} + \frac{M^2(K-n)}{K-n} \,. \tag{3.5}$$

Of course, the physics of the two theories is very different, and this easily explains the few differences of the spectra. The crucial point is that we thus *completely* understand the spectrum at $g = 0$ — and also at large g. It is very interesting that we produce all multi-particle states at $g = 0$, and that only the mass of the lowest state in each set of states with fixed parton number stays finite in the large-coupling limit. In this sense the theory is dominated by the underlying supersymmetry, although the BPS symmetry seems slightly broken.

In Table I we show the masses and the average numbers of partons and fermions in several of the lowest mass states at resolution $K = 9$ for both the bosonic and the fermionic sectors of the theory as well as the Z_2 even and odd sectors. While masses and average parton numbers are identical between bosonic and fermionic sectors, the average fermion numbers are obviously different.

It is very instructive to look at the structure functions for the computed bound states, particularly since they are very QCD-like. We use a standard definition of the structure functions

$$\hat{g}_A(x) = \sum_q \int_0^1 dx_1 \cdots dx_q \delta\left(\sum_{i=1}^q x_i - 1\right) \sum_{l=1}^q \delta(x_l - x)\delta_{A_l}^A |\psi(x_1, \ldots x_q)|^2 \,. \tag{3.6}$$

TABLE I. Average parton and fermion numbers for the lowest 15 states in the Z_2 even and odd sectors at $K = 9$. While masses and average parton numbers are identical in the bosonic and fermionic sectors, the average fermion numbers $\langle n_F\rangle_B$ and $\langle n_F\rangle_F$, respectively, are obviously different.

No.	$M^2_{Z_2=+1}$	$\langle n\rangle$	$\langle n_F\rangle_B$	$\langle n_F\rangle_F$	$M^2_{Z_2=-1}$	$\langle n\rangle$	$\langle n_F\rangle_B$	$\langle n_F\rangle_F$
1	4.2344	2.0543	0.1095	1.1061	9.7197	3.0982	0.2012	1.1945
2	17.3049	4.1229	0.2539	1.2449	26.8372	5.1282	0.2674	1.2572
3	25.6130	2.3138	1.7763	1.3675	29.7822	3.2386	1.8603	1.2995
4	31.8181	4.1052	2.0004	1.4683	34.1192	3.3053	0.5905	1.1407
5	32.0114	2.4724	0.7135	1.1135	39.9376	4.5267	2.0000	1.3423
6	34.4887	3.9241	1.9292	1.3323	40.5369	4.7211	0.6094	1.2480
7	35.4751	3.8048	0.6713	1.1965	42.6751	3.8168	1.9404	2.5317
8	38.1566	6.0990	0.2467	1.2336	43.8869	4.8453	0.5371	1.2022
9	38.3108	3.6778	1.9026	1.3923	44.4106	4.7409	1.9689	1.3118
10	39.3822	4.0123	0.6590	1.2340	46.8487	3.9974	1.9260	1.3716
11	39.5958	2.8320	1.8687	2.3274	48.1737	3.9713	1.8218	1.2551
12	44.5659	3.6840	1.8453	1.3832	49.2253	4.1032	0.5985	1.1508
13	46.2064	3.6026	0.7155	1.1640	49.7721	4.4400	2.1560	1.6061
14	47.5226	2.8625	1.6782	1.2451	51.1264	7.0836	0.1898	1.1802
15	48.3041	4.7750	1.9301	2.6179	51.9753	3.8548	2.0888	1.5990

Here A stands for either a boson or a fermion. The sum runs over all parton numbers q, and the Kronecker delta $\delta^A_{A_l}$ selects partons with matching statistics A_l. The discrete approximation g_A to the structure function $\hat{g}_A$ with harmonic resolutions K is

$$g_A(n) = \sum_{q=2}^{K} \sum_{n_1,\ldots,n_q=1}^{K-q} \delta\left(\sum_{i=1}^{q} n_i - K\right) \sum_{l=1}^{q} \delta_n^{n_l} \delta^A_{A_l} |\psi(n_1,\ldots n_q)|^2 \,. \tag{3.7}$$

The functions $g_A(n)$ are normalized so that summation over the argument gives the average boson or fermion number; their sum is then the average parton number, and we compute these sums as a test. We scale the momentum distribution to the total longitudinal momentum and plot the structure functions as functions of the longitudinal momentum fraction $x = k^+/P^+$ carried by an individual parton. Several structure functions are shown in Figs. 6, 7, and 8.

We analyze the structure functions of the lowest bound states of this theory in some detail. There are, after all, very few first-principles calculations of structure functions for gauge theories. We are particularly interested in the states that have a dominant number and type of particle, which is what one generally refers to as the valence partons. We will look at both the valence and sea structure functions. Unfortunately, we cannot go to high enough resolution here to get the wee parton structure functions at small x. This is not a limitation of our computing power, but rather of our code which is designed for 2+1 dimensional problems. We estimate that a rewritten code could go to resolution 50 with our present computing power.

We have seen that the eigenvalues of the theory converge rapidly, and we therefore expect

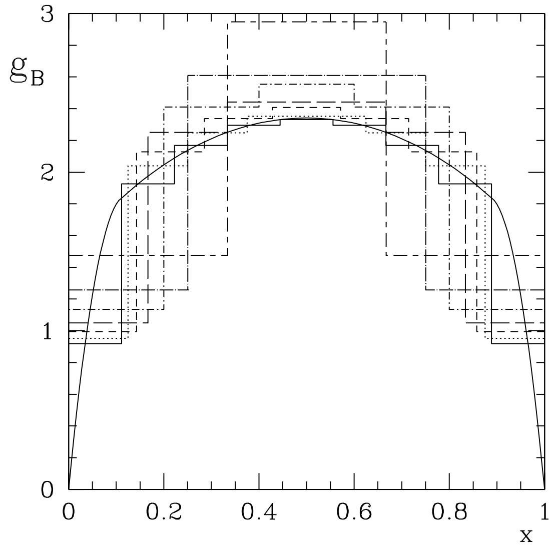


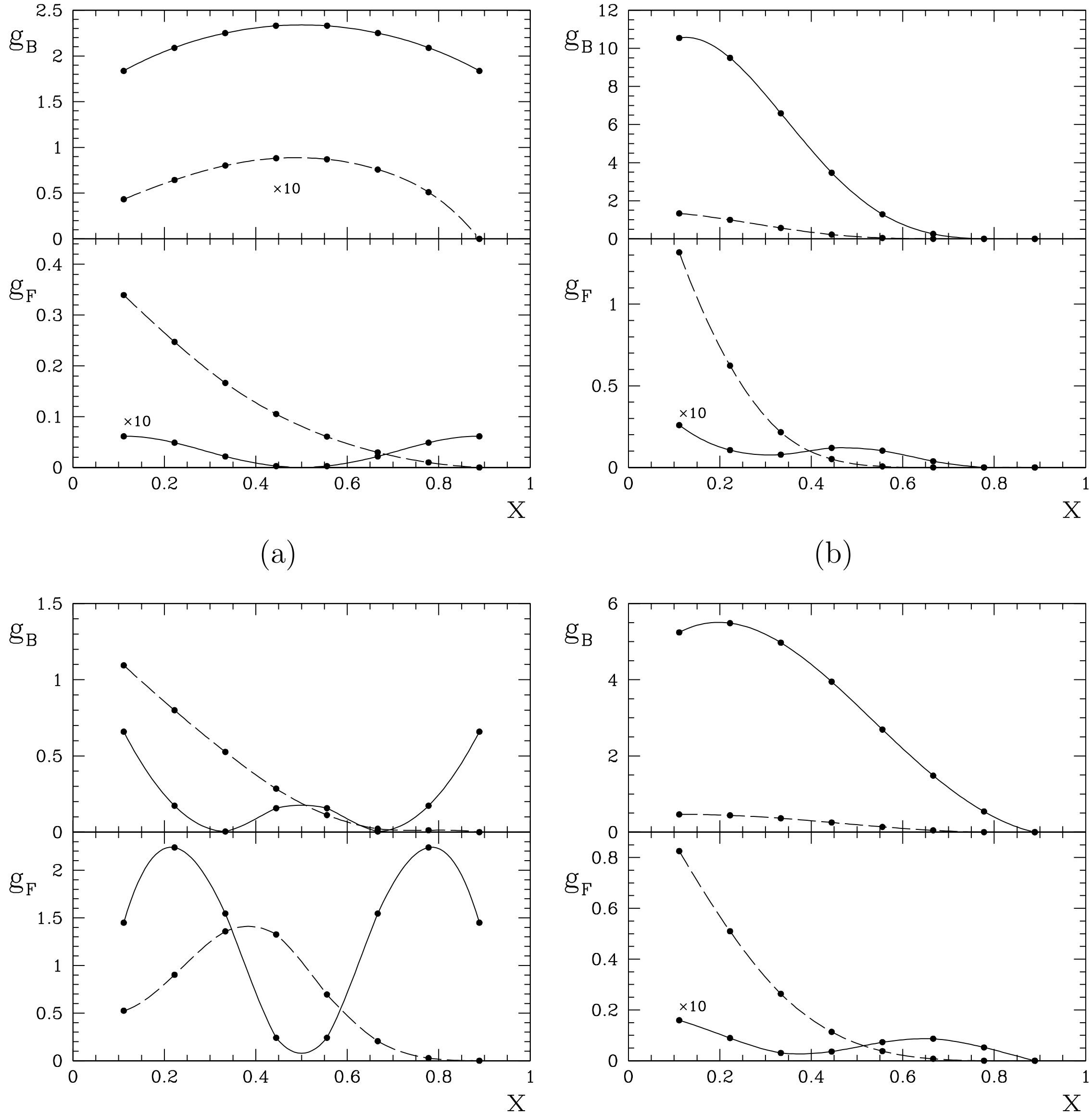
FIG. 6. Convergence of the structure function of the lightest state of the theory as the resolution K is increased from 3 to 9. The smooth solid line is a spline interpolation to the data for $K = 9$ and the conjectured points at $x = 0$ and $x = 1$. The scaled Chern–Simons coupling h is equal to 1.

that the same will be true for the wave functions. We can demonstrate this convergence as follows. We consider the structure function of the lowest mass state in the Z_2 even sector at various values of the resolution K and focus on the valence boson distribution of the lowest boson state. This state is dominantly a two-gluon bound state. For this demonstration we assume that the distributions vanish at $x = 0$ and $x = 1$, which seems sensible for valence partons. We see in Fig. 6 that the structure function appears to converge to a well-defined curve, supporting the notion that the wave function as well as the eigenvalues converge rapidly.

This structure function for the valence gluons (which is also shown in Fig. 7(a)) in the lowest bosonic state is peaked at $x = 0.5$, as one would expect for a two-gluon bound state and interestingly the distribution is quite broad. As we know, in QCD a glueball state will naturally mix with the fermions in the theory. In Fig. 7(a) we see the sea fermion structure function for this state as well as the sea boson structure functions. We find that the fermion distributions tend to peak at low x and are relatively small as compared to the valence distributions. The boson sea distribution, which is primarily from the three-parton component of the wave function, does not peak at small x. The reason is that the small-x component of the three-parton wave function has a nearly equal mixture of fermion pairs and boson pairs with small x. These small-x fermions are seen in the fermion distribution.

The structure functions of the second lowest bound states in the Z_2 even bosonic sector are shown in Fig. 7(b). This is a four-gluon bound state, the analog of what is referred to as an "odd ball" in QCD. The valence-gluon structure function is expected to peak at about $x = 0.25$, but we see that the actual peak is at smaller values of x. The various sea distributions are shown in Fig. 7 and again appear to peak at small x.

The third state in the Z_2 even bosonic sector is a highly mixed state containing a nearly equal mixture of fermions and gluons and does not appear to have the simple valence structure of the lowest two states. In Fig. 7(c) we show separately the distribution of the fermions



(c) (d)

FIG. 7. Structure functions of the lightest three Z_2 even *bosonic* states, with (a) $M^2 = 4.23$, (b) $M^2 = 17.30$, (c) $M^2 = 25.60$, and (d) the lightest Z_2 odd state with $M^2 = 9.72$. These masses are in units of $g^2 N_c/\pi$ at fixed $h = 1$, and the numerical resolution is $K = 9$. The top half of each plot shows the bosonic structure functions, and the bottom half, the fermionic structure functions. Dashed lines represent sea structure functions; solid lines represent structure functions in the two-parton sector, for (a) and (c); in the four-parton sector for (b); and in the three-parton sector for (d). For the purpose of visibility, the following functions have been multiplied by 10: In (a), the fermionic two-parton and bosonic sea structure functions; in (b), the fermionic four-parton structure function; and in (d), the fermionic three-parton structure function.

and bosons. It is not surprising that we find such a mixed state, since in the supersymmetric theory the fermion and boson are treated on an equal footing, but rather it *is* surprising that the lowest states have a valence structure.

It is also interesting to discuss the fermionic states. At first sight it might be surprising that their structure functions, shown in Fig. 8, are so different from those of their bosonic partners. Even more so since they have the same mass and average parton number, guaranteed by supersymmetry. A closer look reveals, however, that the new structures are due to the different combinatorics induced by the "extra" fermionic parton. For example, the valence structure functions of the state with $M^2 = 25.61$ are symmetric around $x = 0.5$ in the bosonic sector, and almost symmetric in the fermionic sector. The asymmetry is clearly induced by the need to have an additional parton for the statistics. For the states with $M^2 = 9.72$ and $M^2 = 17.30$ it seems as though the curves stay more or less the same, except that the fermion valence structure function is greatly enhanced. The lightest state has an interesting valence structure function: It is exactly as probable to find a boson with momentum fraction x, as it is to find a fermion with $1 - x$.

IV. SUMMARY

We have presented an analysis of dimensionally reduced supersymmetric YM theory with a CS term using the numerical method SDLCQ, which exactly preserves the supersymmetry of this theory. We constructed finite dimensional representations of the superalgebra and from them a finite dimensional Hamiltonian which we solved numerically. As we go to higher dimensional representations, the solutions converge rapidly.

¿From these solutions we extracted the properties of the bound states of this theory. We found that the bound states are very different from the bound states of $\mathcal{N} = 1$ SYM in 1+1 dimensions. The bound states of SYM are characterized by their stringy nature, that is a set of states where the states with the most partons have the lowest energy. The CS theory is generally very QCD-like. The states with more partons tend to be more massive, and many of the low-mass states have a valence-like structure. These states have a dominant component of the wave function with a particular number and type of parton. We have found the spectrum of these states and studied it as a function of the CS coupling. We found that, as expected, the CS coupling behaves as an effective mass. As it increases we see that the masses of the bound states generally increase, and the average number of partons of the low-mass bound states tends to decrease.

Also, the SYM theory has an interesting set of massless BPS states. These states are reflected in the CS theory as a set of states whose masses are approximately independent of g and equal to the square of the sum of the CS masses of the partons in the bound state. Since these BPS states are also present in the (2+1)-dimensional SYM theory, we expect to see their reflection in the (2+1)-dimensional CS theory.

We have investigated the structure functions for these bound states and found that they behave in very interesting ways. The distribution for the states with a valence structure are peaked near the inverse of the number of valence partons, and the sea distribution appears to peak at small x in most if not all cases. We also see strongly mixed states with interesting double-humped structure functions. There have been interesting conjectures about structure

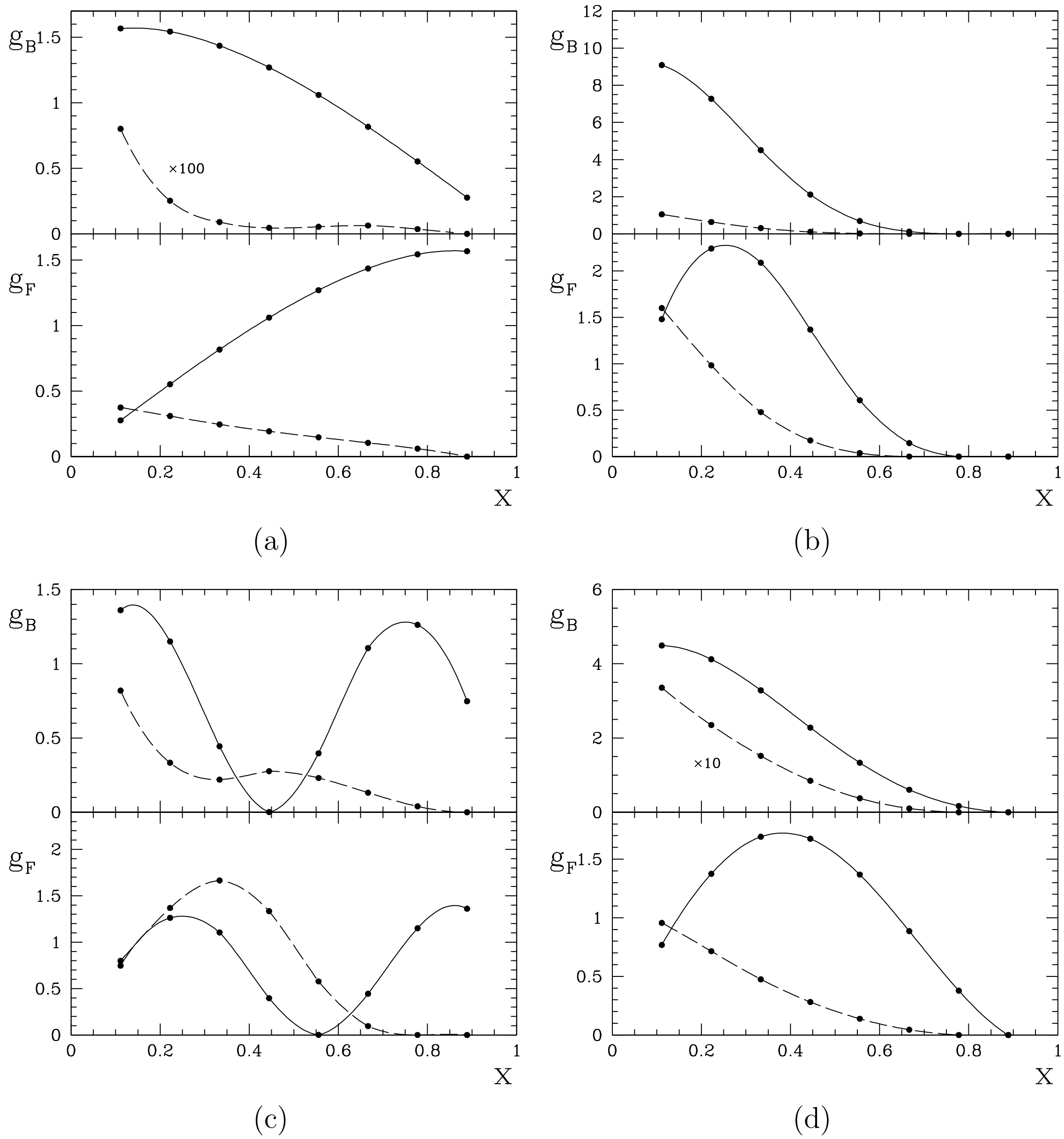


FIG. 8. Same as Fig. 7 but for fermionic states, with only the bosonic sea structure functions in (a) and (d) rescaled by 100 and 10, respectively.

functions of this type in QCD [27], and it is interesting that we actually find such a structure function in the solution of an actual gauge theory.

In summary, the SDLCQ solutions of dimensionally reduced supersymmetric CS theory are very interesting, maybe more interesting, at least with respect to their QCD-like structure, than are the solutions for pure supersymmetric SYM theory. Clearly this provides a strong base and encouragement to move on to 2+1 dimensions.

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REFERENCES

[1] G. V. Dunne, arXiv:hep-th/9902115.

[2] For a nice discussion of the topological properties in the context of an abelian theory, see A. Polychronakos, Ann. Phys. **203**, 231 (1990).

[3] H.-C. Kao, K. Lee, and T. Lee, Phys. Lett. B **373**, 94 (1996), arXiv:hep-th/9506170.

[4] E. Witten, in *The many faces of the superworld*, edited by M.A. Shifman, (World Scientific, Singapore, 2000), p. 156, arXiv:hep-th/9903005.

[5] E. Witten, Nucl. Phys. **B268**, 253 (1986).

[6] L. Susskind, arXiv:hep-th/0101029.

[7] Y. Matsumura, N. Sakai, and T. Sakai, Phys. Rev. D **52**, 2446 (1995).

[8] J. R. Hiller, S. Pinsky and U. Trittmann, Phys. Rev. D **64**, 105027 (2001), arXiv:hep-th/0106193.

[9] J. R. Hiller, S. Pinsky and U. Trittmann, Phys. Rev. D **63**, 105017 (2001), arXiv:hep-th/0101120.

[10] F. Antonuccio, O. Lunin, S. Pinsky , and S. Tsujimaru, Phys. Rev. D **60**, 115006 (1999), arXiv:hep-th/9811254.

[11] O. Lunin and S. Pinsky, Phys. Rev. D **63**, 045019 (2001), arXiv:hep-th/0005282.

[12] S. Pinsky and U. Trittmann, Phys. Rev. D **62**, 087701 (2000), arXiv:hep-th/0005055.

[13] J. R. Hiller, O. Lunin, S. Pinsky and U. Trittmann, Phys. Lett. B **482**, 409 (2000), arXiv:hep-th/0003249.

[14] P. Haney, J. R. Hiller, O. Lunin, S. Pinsky and U. Trittmann, Phys. Rev. D **62**, 075002 (2000), arXiv:hep-th/9911243.

[15] O. Lunin and S. Pinsky, AIP Conf. Proc. **494**, 140 (1999), arXiv:hep-th/9910222.

[16] F. Antonuccio, O. Lunin and S. Pinsky, Phys. Rev. D **59**, 085001 (1999), arXiv:hep-th/9811083.

[17] F. Antonuccio, H. C. Pauli, S. Pinsky and S. Tsujimaru, Phys. Rev. D **58**, 125006 (1998), arXiv:hep-th/9808120.

[18] F. Antonuccio, O. Lunin, S. Pinsky, H. C. Pauli and S. Tsujimaru, Phys. Rev. D **58**, 105024 (1998), arXiv:hep-th/9806133.

[19] F. Antonuccio, O. Lunin and S. Pinsky, Phys. Rev. D **58**, 085009 (1998), arXiv:hep-th/9803170.

[20] F. Antonuccio, O. Lunin and S. S. Pinsky, Phys. Lett. B **429**, 327 (1998), arXiv:hep-th/9803027.

[21] D. Kutasov, Nucl. Phys. B **414**, 33 (1994).

[22] K. Hornbostel, Ph.D. thesis, SLAC Report No. 333, 1988.

[23] H.-C. Pauli and S.J. Brodsky, Phys. Rev. D **32**, 1993 (1985); **32**, 2001 (1985).

[24] S.J. Brodsky, H.-C. Pauli, and S.S. Pinsky, Phys. Rep. **301**, 299 (1998), arXiv:hep-ph/9705477.

[25] J.S. Rozowsky and C.B. Thorn. Phys. Rev. Lett. **85**, 1614 (2000), arXiv:hep-th/0003301.

[26] U. Trittmann, Nucl. Phys. B **587**, 311 (2000), arXiv:hep-th/0005075; arXiv:hep-th/0110058.

[27] V.L. Chernyak and A.R. Zhitnitsky, Phys. Rep. **112**, 173 (1984).